CONVERGENCE OF YANG-MILLS-HIGGS FLOW FOR TWIST HIGGS PAIRS ON RIEMANN SURFACES

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ABSTRACT. We consider the gradient flow of the Yang-Mills-Higgs functional of twist Higgs pairs on a Hermitian vector bundle (E, H_0) over a Riemann surface X. It is already known the gradient flow with initial data (A_0, ϕ_0) converges to a critical point (A_∞, ϕ_∞) of this functional. Using a modified Chern-Weil type inequality, we prove that the limiting twist Higgs bundle $(E, d''_{A_\infty}, \phi_\infty)$ is given by the graded twist Higgs bundle defined by the Harder-Narasimhan-Seshadri filtration of the initial twist Higgs bundle (E, d''_{A_0}, ϕ_0), generalizing Wilkin's results for untwist Higgs bundle.

1. INTRODUCTION

Higgs bundle originates from Hitchin's reduction of self-dual equation on $\mathbb{R}^4$ to Riemann surface(cf. [Hit87]), constituted by a holomorphic vector bundle $E \to X$, and a holomorphic (1,0)-form ϕ taking value in $End(E)$. If the base manifold X is a smooth Riemann surface, it is equivalent to say $\phi \in H^0(X, End(E) \otimes K)$, where K is the canonical line bundle of X. This suggests us that K can be replaced by any line bundle. The definition of twist Higgs bundle follows

Definition 1.1. *A twist Higgs bundle is a pair (E, ϕ) where E is a rank n holomorphic vector bundle over a complex manifold X, $\phi \in \Omega^{1,0}(End(E) \otimes L)$ is the holomorphic Higgs field twisted with line bundle L, where L is any fixed holomorphic line bundle.*

To emphasis on the holomorphic structure, Higgs bundle also can be denoted as $(E, \bar{\partial}, \phi)$. If we take L to be the trivial line bundle, then it becomes the usual Higgs bundle. For simplicity, in this article, Higgs bundle means twist Higgs bundle, and we will specify the usual Higgs bundle as untwist Higgs bundle.

Twist Higgs bundles share lots of tributes with the untwist Higgs bundles. One can define stability on twist Higgs bundle and the Hitchin-Kobayashi correspondence still holds(cf. [BDGPW95]). The twist Higgs bundle also admit the so called Harder-Narasimhan-Seshadri(short as HNS) filtration(cf. [HT03]). While, there are someting different, for example, the moduli space of twist Higgs bundle is just a coarse moduli(cf. [Nit91]) rather than a fine moduli as the usual Higgs bundle(cf. [Hit87]).

Restrict ourselves to the case X is a Riemann surface, fix a C^∞ complex vector bundle E of rank n with a Hermitian metric H and a holomorphic line bundle L with Hermitian metric h. Let $\mathcal{A}$ denote the space of connections on E compatible with the metric. Notice that $\mathcal{A}$ is isomorphic to the space $\mathcal{A}^{0,1}$, the space of holomorphic structures on E. A pair $(A, \phi) \in \mathcal{A} \otimes \Omega^{1,0}(End(E) \otimes L)$ is called a Higgs pair if $d_A'' \phi = 0$ is satisfied, where d_A'' is the naturally induced covariant derivative on $\Omega^{1,0}(End(E) \otimes L)$. Thus each Higgs pair will endow E a structure of Higgs bundle (E, d_A'', ϕ). In [Wil06], Wilkin studied the Yang-Mills-Higgs(short as YMH) flow of untwist Higgs pair over Riemann surface, proved that the flow converges to the graded object associated to the Harder-Narasimhan-Seshadri filtration. In this article, we generalize the result to twist Higgs pair. The convergence of the flow is analogous to Wilkin's case, or can be viewed as a special case of Yue Wang and Xi Zhang's work(cf. [WZ11]). So we focus on the asymptotic behavior of the heat flow for Higgs pairs. Our first main theorem asserts that the gradient flow preserve the Harder-Narasimhan(short as HN) type. The key point in the proof is the Chern-Weil formula(cf. [Sim88]) for subbundle S defined by projection π

$$deg(S) = \frac{1}{2\pi} \int_X (Tr(\sqrt{-1}\Lambda F_A \pi) - |d_A'' \pi|^2) dvol.$$

Using this formula, we modify the method in [DW04] to control the degree of twist Higgs sub-bundle, and show that the HN type is non-decreasing along the flow. Following the idea of Atiyah and Bott(cf. [AB83]), Wilkin employed all the convex invariant function on the Lie algebra to identify the HN type of the Higgs bundle. His method is still available for twist Higgs bundle on Riemann surface, but we take the method of Daskalpoulos and Wentworth(cf. [DW04]), using merely a subclass of convex functional the so called weighted YMH functional, which is easy to be generalized to the Higgs bundle on Kähler surface(cf. [LZ11]). Combine the non-increasing of the weighted YMH functional and the non-decreasing of the HN type along the gradient

flow, then use the so-called approximate Hermitian structure to eliminate the possibility of jumping phenomenon, we get our first main theorem.

Theorem 1.2. *Let (A_t, ϕ_t) be a smooth solution of the gradient flow on the Hermitian vector bundle (E, H) with initial condition (A_0, ϕ_0) and (A_∞, ϕ_∞) be the limit. Then the Harder-Narasimhan type of $(E, d''_{A_\infty}, \phi_\infty)$ is the same as that of (E, d''_{A_0}, ϕ_0).*

Following Donaldson[Don85], we constructs a nontrivial holomorphic map $(E, d''_{A_0}, \phi_0) \to (E, d''_{A_\infty}, \phi_\infty)$. With such a map in hand, one may then apply the basic principle that a nontrivial holomorphic map between stable bundles of the same rank and degree must be an isomorphism. Denote the graded object associated to the HNS filtration by $Gr^{hns}(E, d''_{A_0}, \phi_0)$, there is the second main theorem

Theorem 1.3. *The Higgs bundle $(E, d''_{A_\infty}, \phi_\infty)$ is holomorphically isomorphic to the graded object $Gr^{hns}(E, d''_{A_0}, \phi_0)$.*

One may make further investigation on the moduli space of twist Higgs bundle over Riemann surface, getting the stratification structure according to the HN type. The discussion is totally parallel with the untwist case in [Wil06]. Recently, there are several excellent works about the convergence of the Yang-Mills flow(cf. [Sib12],[CJ12]) on manifold with dimension greater than two. We hope that our results on twist Higgs bundle could be generalized to higher dimension.

This article is organized as follows. In section 2, we collect some preliminary material about Higgs bundle, such as the Hitchin's equation, stability, Hitchin-Kobayashi correspondence, HNS filtration, the Yang-Mills-Higgs functional of Higgs pair and its gradient flow. In section 3, we focus on the Harder-Narasimhan type of the limit of the gradient flow. Using tools like weighted YMH functional and approximate critical Hermitian structure, we prove the first main theorem. Finally, in section 4, we get the proof for our second main theorem.

2. Preliminary

2.1. **Higgs bundle.** Fix a Hermitian metric h on the holomorphic line bundle L once for all, there is a unique Chern connection compatible with h and the holomorphic structure.

Endow Hermitian metric H on E, denote $A = (\bar{\partial}, H)$ the connection 1-form of Chern connection respect to H s.t. $d''_A = \bar{\partial}$, and $F_{(\bar{\partial}, H)}$ the curvature two form. Combining the fixed connection on L, there is induced connection on $End(E) \otimes L$, still denoted as d_A. If $\phi = \Phi dz \otimes s$, $\Phi \in \Gamma(End(E))$, $s \in \Gamma(L)$, then ϕ^{*H} is set to be $\Phi^{*H} d\bar{z} \otimes h(s)$, here Φ^{*H}

is the adjoint of Φ under H, $h(s)$ is the section of L^* defined by $h(s, \ )$. Moreover, we define $[\phi, \phi^{*H}] \in End(E)$ as the Lie bracket extended to $End(E) \otimes L$ and $End(E) \otimes L^*$ valued 1-form, means $\phi\phi^{*H} + \phi^{*H}\phi$, where the contraction of L with L^* is taken place at the same time.

There is the so called Hitchin's equation on the Hermitian metric H over twist Higgs bundle $(E, \bar{\partial}, \phi)$.

$$(2.1) \qquad \sqrt{-1}\Lambda_\omega(F_{(\bar{\partial}, H)} + [\phi, \phi^{*H}]) = \mu Id_E$$

where ω is any fixed Kähler form(the equation is conformal invariant, independent on the metric on X, so sometimes we neglect the subscript ω). For $[\phi, \phi^{*H}]$ is always traceless, $\mu = \frac{degE}{rankE}$ is the slope of the vector bundle.

A natural question is when a Higgs bundle admits a solution of Hitchin's system. More preciously, for a fixed holomorphic bundle $(E, \bar{\partial})$ and holomorphic $\phi \in \Omega^{1,0}(End(E) \otimes L)$, find a Hermitian metric H, s.t. the Chern connection $(\bar{\partial}, H)$ and ϕ, ϕ^{*H} satisfying the equation. This turns out to closely relate to the so called stability. The Hitchin-Kobayashi correspondence is totally parallel with the untwist situation,

Definition 2.1. *A Higgs bundle $(E, \bar{\partial}, \phi)$ is called stable(semi-stable) if for any ϕ invariant[1] holomorphic subbundle F, $\frac{degF}{rankF} < (\leq)\frac{degE}{rankE}$. Moreover, the Higgs bundle is called polystable if $(E, \phi) = (E_1, \phi_1) \oplus \cdots \oplus (E_r, \phi_r)$, here $\phi_i \in \Omega^{1,0}(End(E_i) \otimes L)$ and (E_i, ϕ_i) are all stable and with same slope.*

and

Theorem 2.2 ([Hit87],[Sim88])**.** *A Higgs bundle $(E, \bar{\partial}, \phi)$ admit a Hermitian metric H satisfying the Hitchin equation if and only if it is polystable.*

Readers can refer to [Hit87],[Sim88] for the proof of the untwist case, and [BDGPW95] for the twist case.

If we restrict ourselves to the untwist Higgs bundle, consider the new $GL(n, \mathbb{C})$ connection $\nabla_{(\bar{\partial}, \phi, H)} = d_A + \phi + \phi^{*H}$, denote its curvature by $R_{(\bar{\partial}, \phi, H)}$ to distinguish with $F_{(\bar{\partial}, H)}$. Although $\nabla_{(\bar{\partial}, \phi, H)}$ is not a metric connection to H,

$$R_{(\bar{\partial}, \phi, H)} = \nabla^2_{\bar{\partial}, \phi, H} = d_A^2 + d_A(\phi + \phi^{*H}) + [\phi, \phi^{*H}]$$
$$= F_{(\bar{\partial}, H)} + [\phi, \phi^{*H}] + d_A''\phi + d_A'\phi^{*H}.$$

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[1]means that $\phi(F) \subset F \otimes L \otimes K$

This means that if H is a solution to Hitchin's equation of the untwist Higgs bundle, then the traceless part of the curvature satisfies $R^{\perp}_{(\bar{\partial},\phi,H)} = 0$.

In the twist case, we can not produce the non-unitary connection from the Higgs data, but the behavior of $F_{(\bar{\partial},H)} + [\phi, \phi^{*H}]$ is somehow similar to $R_{(\bar{\partial},\phi,H)}$, so we denote $F_{(\bar{\partial},H)} + [\phi, \phi^{*H}]$ as $\Theta_{(\bar{\partial},\phi,H)}$ for the twist Higgs bundle to prevent ambiguity. Sometimes we will abbreviate some subscripts as Θ_H if only the metric H varies.

2.2. Harder-Narasimhan and Seshadri filtrations.

Given arbitrary twist Higgs bundle $(E, \bar{\partial}, \phi)$, it may not be stable or semitable, but we have

Lemma 2.3. *Let $(E, \bar{\partial}, \phi)$ be a twist Higgs bundle, then there is a unique filtration, called Harder-Narasimhan filtration of E by ϕ-invariant holomorphic subbundles $0 = E_0 \subset E_1 \subset \cdots \subset E_r = E$, s.t. $F_i = E_i/E_{i-1}$ is Higgs semistable with respect to the quotient Higgs field $\tilde{\phi}_i \in \Omega^{1,0}(End(F_i) \otimes L)$ for all i and $\mu(F_1) > \cdots > \mu(F_r)$.*

And the n-tuple $\overrightarrow{\mu} = (\overbrace{\mu_1, \cdots, \mu_1}^{rankE_1}; \overbrace{\mu_2, \cdots, \mu_2}^{rankE_2} \cdots \overbrace{\mu_r, \cdots, \mu_r}^{rankE_r})$ is called the type of the HN filtration. Similarly,

Lemma 2.4. *Let $(E, \bar{\partial}, \phi)$ be a semi-stable Higgs bundle, then there is a filtration, called Seshadri filtration of E by ϕ-invariant holomorphic subbundles $0 = E_0 \subset E_1 \subset \cdots \subset E_s = E$, s.t. $Q_i = E_i/E_{i-1}$ is Higgs stable for all i.*

Notice that for Q_i may have same slope, Seshadri filtration is not unique. Combining above two filtrations together, we have

Proposition 2.5. *Let $(E, \bar{\partial}, \phi)$ be a Higgs bundle, then there is a double filtration $\{E_{i,j}\}$ of E called ϕ-invariant Harder-Narasimhan-Seshadri filtration, s.t. $\{E_i\}_{i=1}^r$ is the HN filtration, and $\{E_{i,j}\}_{j=1}^{s_i}$ is a Seshadri filtration of E_i/E_{i-1}.*

The associated graded object

$$Gr^{HNS}(E, \bar{\partial}, \phi) = \oplus_{i=1}^r \oplus_{j=1}^{s_i} Q_{i,j},$$

where $Q_{i,j} = E_{i,j}/E_{i,j-1}$, is uniquely determined by the isomorphism class of $(E, \bar{\partial}, \phi)$. It is easy to see that $Gr^{HNS}(E, \bar{\partial}, \phi)$ is not gauge equivalent to $(E, \bar{\partial}, \phi)$ except itself is stable. This provides us an algebraic way to split a Higgs bundle into a direct sum of stable Higgs bundles.

2.3. Higgs-Yang-Mills flow.

On the other hand, rather than fixing the Higgs bundle (E, ∂, ϕ) to find the Hermitian Yang-Mills metric H, we fix the C^∞ bundle E and a Hermitian metric H on it, to find a (integrable)connection compatible with the metric and a $\phi \in \Omega^{1.0}(End(E) \otimes L)$, such that, firstly, E is a holomorphic vector bundle[2] with holomorphic structure d''_A; Secondly, ϕ is holomorphic under $\bar{\partial} = d''_A$, i.e. a Higgs field, thus the Higgs pair (A, ϕ) makes (E, d''_A, ϕ) a Higgs bundle. Thirdly, H satisfies the Hitchin's equation. Summing up, for fixed H, there is a equation on the Higgs pair (A, ϕ)

$$(2.2) \qquad \begin{cases} F_A + [\phi, \phi^{*H}] = -\sqrt{-1}\mu I d_E \omega \\ d''_A \phi = 0 \end{cases}$$

we still call it the Hitchin's equation. In this turn, we denote $F_A + [\phi, \phi^{*H}]$ as $\Theta_{(A,\phi)}$, and we will omit the subscript if it does not cause confusion. The solutions of this equation can be interpreted in Morse theory. Consider the Yang-Mills-Higgs functional on (A, ϕ) restricting to the level set $d''_A \phi = 0$

$$(2.3) \quad \mathrm{YMH}(A, \phi) = ||F_A + [\phi, \phi^{*H}]||^2 = \int |\Lambda_\omega(F_A + [\phi, \phi^{*H}])|^2 \mathrm{d}vol,$$

the solutions of Hitchin equation is the local minimum, a subclass of the critical points of this functional. To find all the critical points, we consider the associated gradient flow

$$\begin{cases} \frac{\partial A}{\partial t} = *d_A * (F_A + [\phi, \phi^{*H}]) = -d_A^* \Theta \\ \frac{\partial \phi}{\partial t} = *[\phi, *(F_A + [\phi, \phi^{*H}])] = *[\phi, *\Theta] \end{cases}$$

Since both the holomorphic structure and Hermitian metric are fixed, the connection A is totally determined by its $(0, 1)$ part A'', the first equation is equivalent to

$$\frac{\partial A''}{\partial t} = *d''_A * (F_A + [\phi, \phi^{*H}]) = -d''^*_A \Theta.$$

Before solving this evolution equation system for any initial data, we should notice that $*$ acting on 1-form amounts to multiplying the complex number i, and

$$(2.4) \qquad \begin{aligned} \frac{\partial d''_A \phi}{\partial t} &= \frac{\partial}{\partial t}(d''\phi + [A'', \phi]) = d''_A \frac{\partial \phi}{\partial t} + [\frac{\partial A''}{\partial t}, \phi] \\ &= i[d''_A * \Theta, \phi] + i d''_A[\phi, *\Theta] \\ &= i((d''_A \Theta)\phi + \phi d''_A \Theta + d''_A(\phi\Theta) - d''_A(\Theta\phi)) \\ &= i((d''_A \phi)\Theta - \Theta(d''_A \phi)) = i[d''_A \phi, \Theta] \end{aligned}$$

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[2]This always holds on Riemann surface

Thus the holomorphicity of Higgs field $d_A''\phi = 0$ is preserved by the gradient flow, it makes sense to restrict on the level set $d_A''\phi = 0$ to solve the Cauchy problem of the gradient flow.

To get the existence and convergence properties of this gradient flow, by Simpson[Sim88], one fixes (A_0, ϕ_0), letting H change along the following heat equation

$$(2.5) \qquad\qquad H^{-1}\frac{\partial H}{\partial t} = -i\Lambda\Theta_H^\perp.$$

If $H(t)$ is the solution for this equation, then there is a gauge transformations $g(t)$ determined by $H(t)$, s.t. $(A(t), \phi(t)) = (g(t) \cdot A_0, g(t) \cdot \phi_0)$ will be a solution to Equation (2.4)(the explicit expression of the gauge transformation can be found in [Wil06]). In the untwist case, Simpson had proved that solution to Equation (2.5) exists for all time and depends continuously on the initial condition $H(0)$. The twist case can be viewed as a special case of [WZ11]. Via the equivalence of above heat flow and the gradient flow of YMH, Wilkin(cf. [Wil06]) proved the following properties of the solution to Equation (2.4)(the proof in twist case is identical).
- Existence for all time and uniqueness.
- Convergence modulo gauge transformation.
- Convergence without gauge transformation.
- Continuous dependence on initial condition for any fixed $T < \infty$
in the H^k norm, for any $k \in \mathbb{N}$.

If the initial data (A_0, ϕ_0) define a stable Higgs bundle (E, d_{A_0}'', ϕ_0), then the limit (A_∞, ϕ_∞) will satisfy Equation (2.2), i.e. there is a gauge transformation relate (A_0, ϕ_0) to (A_∞, ϕ_∞). Without any stable assumption on the initial data, (A_∞, ϕ_∞) may not be a solution of the Hitchin's equation. There should be a precise description of the limit.

Proposition 2.6. *Let (A, ϕ) be a critical point of the YMH functional, then there is an ϕ-invariant orthogonal splitting $(E, d_A'', \phi) = \oplus_{i=1}^l (E_i, d_{A_i}'', \phi_i)$, s.t.*

$$\sqrt{-1}\Lambda\Theta_i = \mu_i Id_{E_i}$$

where $\Theta_i = F_{A_i} + [\phi_i, \phi_i^{*H}]$ and $\mu_i = \mu(E_i)$.

We only sketch the proof. The critical points of Equation (2.3) satisfying the Euler-Lagrange equations $d_A * \Theta = 0$ and $[\phi, *\Theta] = 0$. The first equation implies the eigenvalues of $*\Theta$ are all constant hence inducing a splitting of the vector bundle. While second equation shows that this splitting is ϕ-invariant. So this analytic limit splits into direct sum of polystable Higgs bundle.

Recall for a twist Higgs bundle $(E, \bar{\partial}, \phi)$, there is a graded Higgs bundle obtained via HNS filtration. Endow any Hermitian metric H on this twist Higgs bundle, denote the compatible Chern connection as A. Forget the holomorphic structure on E, consider the YMH flow for this twist Higgs pair (A, ϕ) on the C^∞ bundle E, there is also a split bundle at the limit. We want to show that this two kinds of splitting coincide. The proof is divided into two steps. We first show the gradient flow keep the HN type. Secondly, we show the limit of the gradient flow must be the graded object defined by the HNS filtration.

3. Harder-Narasimhan type of the limit

The solution of the YMH flow in finite time equals to a gauge transformation, so the jumping phenomenon of the HN type only takes place at the limit. We try to relate the HN type with the weighted YMH functionals, and use these functionals to identify the HN type.

3.1. YMH functional and HN type.
Recall some basic facts about the YMH functional and HN type without proof.

Proposition 3.1. *Let (A_t, ϕ_t) be a solution of Equation (2.4), then*

$$\frac{\partial}{\partial t}|\Lambda\Theta|^2 + \triangle_A|\Lambda\Theta|^2 \leq 0.$$

where $\triangle_A$ is the Hodge Laplace of d_A. Furthermore, integrate the above expression, there is $\frac{d}{dt}\|\Theta\|^2 = -2\|d_A^*\Theta\|^2 \leq 0$, i.e. $t \to YMH(A_t, \phi_t)$ is non-increasing.

The proof of the untwist case follows form [Wil06], the twist case is a special case of [WZ11]. By the convergence of the YMH flow, $\Lambda R_t \xrightarrow{L^p} \Lambda R_\infty$, there is

Lemma 3.2.

$$\lim_{t\to\infty} YMH(A_t, \phi_t) = YMH(A_\infty, \phi_\infty).$$

In order to compare different HN type, define a partial order of the n-tuple $\overrightarrow{\mu} = (\mu_1, \cdots, \mu_n)$, $\mu_1 \geq \cdots \geq \mu_n$. For the Chern class is fixed, we only need to take care the case $\sum_{i=1}^n \mu_i = \sum_{i=1}^n \lambda_i$. We call $\overrightarrow{\mu} \leq \overrightarrow{\lambda}$ if $\sum_{j\leq k} \mu_j = \sum_{j\leq k} \lambda_j$ for all $k = 1, \ldots, n$. As we know, if E admit a critical twist Higgs pair, then it splits. Abuse the notation, let $\overrightarrow{\mu}$ denote the split bundle, then $YMH(\overrightarrow{\mu}) = 2\pi \sum_{i=1}^n \mu_i^2$. It is easy to verify $\overrightarrow{\mu} \leq \overrightarrow{\lambda}$ implying $YMH(\overrightarrow{\mu}) \leq YMH(\overrightarrow{\lambda})$. In the next, we study how the HN type changes along the gradient flow. We need an algebraic lemma.

Lemma 3.3. *Let $(E, \bar{\partial}, \phi)$ be a twist Higgs bundle and S be a ϕ invariant subbundle. Endow a Hermitian metric on E, let $\pi = \pi^* = \pi^2$ denote the orthogonal projection onto the subbundle S. Then*

$$Tr([\Phi, \Phi^*]\pi) = |[\phi, \pi]|^2.$$

where the inner product $|[\phi, \pi]|^2$ is defined to be $Tr([\phi, \pi][\phi, \pi]^{*H})$ after contraction the section of L and L^* by the fix Hermitian metric h.

Proof. Compute it straight forward.

$$\begin{aligned}|[\phi, \pi]|^2 &= Tr([\phi, \pi][\phi, \pi]^{*H}) = Tr((\phi\pi - \pi\phi)(\pi\phi^{*H} - \phi^{*H}\pi)) \\ &= Tr(\phi\pi\pi\phi^{*H} - \phi\pi\phi^{*H}\pi - \pi\phi\pi\phi^{*H} + \pi\phi\phi^{*H}\pi)\end{aligned}$$

by the acyclicity of the trace, there is $-\phi\pi\phi^{*H}\pi = \pi\phi^{*H}\pi\phi$ thus

$$-\phi\pi\phi^{*H}\pi - \pi\phi\pi\phi^{*H} = [\pi\phi^{*H}, \pi\phi]$$

which is always trace free. Still by the acyclicity,

$$|[\phi, \pi]|^2 = Tr(-\phi^{*H}\phi\pi\pi + \phi\phi^{*H}\pi\pi) = Tr([\phi, \phi^{*H}]\pi)$$

$\square$

By Simpson([Sim88]), the Chern-Weil formula reads

$$\begin{aligned}deg(S) &= \frac{1}{2\pi} \int_X Tr(\sqrt{-1}\Lambda F_A \pi) - |d_A''\pi|^2 dvol \\ (3.1) \quad &= \frac{1}{2\pi} \int_X Tr(\sqrt{-1}\Lambda\Theta\pi) - Tr(\sqrt{-1}\Lambda[\phi, \phi^{*H}]\pi) - |d_A''\pi|^2 dvol \\ &= \frac{1}{2\pi} \int_X Tr(\sqrt{-1}\Lambda\Theta\pi) dvol - \frac{1}{2\pi}||d_A''\pi||^2 - \frac{1}{2\pi}||[\phi, \pi]||^2\end{aligned}$$

Proposition 3.4. *Denote the unitary gauge group of E with fixed Hermitian metric H by $\mathfrak{u}(E)$. Let $(A_j, \phi_j) = g_j \cdot (A_0, \phi_0)$ be a sequence of complex gauge equivalent Higgs structure and S be a ϕ_0-invariant holomorphic subbundle of (E, d_{A_0}'', ϕ_0) with rank r. Suppose $\sqrt{-1}\Lambda R_j \xrightarrow{L^1} \mathfrak{a}$, where $\mathfrak{a} \in L^1(\sqrt{-1}\mathfrak{u}(E))$, and that the eigenvalues $\lambda_1 \geq \cdots, \geq \lambda_n$ of $\mathfrak{a}$(counted with multiplicities) are constant. Then $deg(S) \leq \sum_{i \leq r} \lambda_i$.*

Proof. Let $\pi_j : E \to g_j(s)$ denote the orthogonal projection. By above Chern-Weil formula

$$
\begin{aligned}
deg(S) =& \frac{1}{2\pi} \int_X Tr(\sqrt{-1}\Lambda\Theta_j\pi_j)dvol - \frac{1}{2\pi}||d''_{A_j}\pi_j||^2 - \frac{1}{2\pi}||[\phi_j, \pi_j]||^2 \\
\leq& \frac{1}{2\pi} \int_X Tr(\sqrt{-1}\Lambda\Theta_j\pi_j)dvol \\
=& \frac{1}{2\pi} \int_X Tr(\mathfrak{a}\pi_j)dvol + \frac{1}{2\pi} \int_X (Tr(\sqrt{-1}\Lambda\Theta_j - \mathfrak{a})\pi_j)dvol
\end{aligned}
$$

Still by linear algebra(cf. the material under the proof of Lemma 2.20 in [DW04]), $Tr(\mathfrak{a}\pi_j) \leq \sum_{i\leq r} \lambda_i$. Let $j \to \infty$, the last term tends to zero ,finishing the proof. $\qquad\square$

Remark: Recall that in the untwist case, Simpson use the connection $D'' = d''_A + \phi$(this is not the (0,1) component ∇'' of the non-unitary connection ∇). The Chern-Weil formula reads

$$
deg(S) = \frac{1}{2\pi} \int_X Tr(\sqrt{-1}\Lambda R - |D''\pi|^2)dvol.
$$

Notice that $|D''\pi|^2 = |d''_A\pi|^2 + |\phi\pi|^2$ for d''_A is a (0,1)-form and ϕ is a (1,0)-form. So the operator D'' is in effect split and we can threat them independently, this is why the results in untwist case can be transported to the twist case(Reader could also refer to [Wil06] for the symplectic geometry interpretation).

Recall the partial ordering of HN types of Higgs bundle $(E, \bar\partial, \phi)$, by the induction on the length of the HN filtration(cf. [DW04]), we have:

Proposition 3.5. *Let (A_t, ϕ_t) be the solution along the YMH flow on a bundle (E, H) of rank n with limit (A_∞, ϕ_∞). Let $\overrightarrow{\mu}_0 = (\mu_1, \ldots, \mu_n)$ be the HN type of (E, d''_{A_0}, ϕ_0), and let $\overrightarrow{\lambda}_\infty = (\lambda_1, \ldots, \lambda_n)$ be the type of $(E, d''_{A_\infty}, \phi_\infty)$. Then $\overrightarrow{\mu}_0 \leq \overrightarrow{\lambda}_\infty$.*

This is equivalent to say that the HN type is non-decreasing. Recall that YMH functional is non-increasing along the gradient flow, we get following easy corollary generalizes a result in [AB83] to Higgs bundle:

Corollary 3.6. *Let $\overrightarrow{\mu}$ be the HN type of $(E, \bar\partial, \phi)$. For any Hermitian metric H, denote A the unitary connection, then $YMH(A, \phi) \geq 2\pi \sum_{i=1}^n \mu_i^2$, and the equality holds iff H is the split Hermitian Yang-Mills metric.*

This corollary asserts that the HN type can be viewed as a lower bound of the YMH functional.

3.2. **Weighted YMH functionals.** Notice that $\mathrm{YMH}(\overrightarrow{\mu}) = \mathrm{YMH}(\overrightarrow{\lambda})$ is only a necessary condition for $\overrightarrow{\mu} = \overrightarrow{\lambda}$. To distinguish different HN types, we need more functionals. On Riemann surface, people often use convex functionals to detect the HN type, for instance [AB83] the vector bundle case and [Wil06] the Higgs bundle case. On Kähler surface, Daskalopoulos and Wentworth([DW04]) restrict themselves to a subclass of convex functionals, namely weighted Yang-Mills functionals. Here we follow their idea, apply this method to the Higgs case(see also [LZ11]). Let $\mathfrak{u}(n)$ denote the Lie algebra of the unitary group $U(n)$. Fix a real number $\alpha \geq 1$. Then for $\mathfrak{a} \in \mathfrak{u}(n)$, a skew hermitian matrix with eigenvalues $\sqrt{-1}\lambda_1, \ldots, \sqrt{-1}\lambda_n$, let $\psi_\alpha(\mathfrak{a}) = \sum_{j=1}^n |\lambda_j|^\alpha$. By Prop. 12.16 in [AB83], ψ_α is a convex function on $\mathfrak{u}(n)$. Moreover, for a given number N, define:

$$\mathrm{YMH}_{\alpha,N}(A, \phi) = \int_X \psi_\alpha(\Lambda\Theta + \sqrt{-1}NId_E)dvol.$$

Take the convention $\mathrm{YMH}_\alpha(A, \phi) = \mathrm{YMH}_{\alpha,0}(A, \phi)$, and notice that $\mathrm{YMH} = \mathrm{YMH}_2$ is the ordinary YMH functional. We make a slight abuse of notation, setting

$$\mathrm{YMH}_{\alpha,N}(\overrightarrow{\mu}) = \mathrm{YMH}_\alpha(\overrightarrow{\mu} + N) = 2\pi\psi_\alpha(\sqrt{-1}(\overrightarrow{\mu} + N))$$

where $\overrightarrow{\mu} + N = (\mu_1 + N, \ldots, \mu_n + N)$ is identified with the diagonal matrix $diag((\mu_1 + N, \ldots, \mu_n + N)$.

Following lemma reveal the connection between weighted YMH functional and the approximate critical Hermitian structure will be studied in next subsection.

Lemma 3.7. *The functional* $\mathfrak{a} \to (\int_X \psi_\alpha(\mathfrak{a})dvol)^{\frac{1}{\alpha}}$ *defines a norm on* $L^\alpha(\mathfrak{u}(E))$ *which is equivalent to the* L^α *norm* $(\int_X (-Tr\mathfrak{a} \cdot \mathfrak{a}^*)^{\frac{\alpha}{2}} dvol)^{\frac{1}{\alpha}}$.

Proof.

$$\frac{1}{C}(\sum_{i=1}^n |\lambda_i|^2)^{\frac{\alpha}{2}} \leq \frac{1}{C}(\sum_{i=1}^n |\lambda_i|)^\alpha \leq \sum_{i=1}^n |\lambda_i|^\alpha \leq C(\sum_{i=1}^n |\lambda_i|)^\alpha \leq C'(\sum_{i=1}^n |\lambda_i|^2)^{\frac{\alpha}{2}}.$$

$\square$

Now we focus on the relation between the weighted YMH functional and the HN type. Similar with the usual YMH functional, we have,

Proposition 3.8. *Let* (A_t, ϕ_t) *be a solution of the gradient flow. Then for any* $\alpha \geq 1$ *and any* N, $t \to YMH_{\alpha,N}(A_t, \phi_t)$ *is nonincreasing.*

Proposition 3.9. *Let (A_∞, ϕ_∞) be a limit of (A_t, ϕ_t), where (A_t, ϕ_t) is a solution to Equation (2.4). Then for any $\alpha \geq 1$ and any N, $\lim_{t \to \infty} YMH_{\alpha,N}(A_t, \phi_t) = YMH_{\alpha,N}(A_\infty, \phi_\infty)$.*

The proof is parallel with the proof in [DW04] for the vector bundle case. The key point of introducing such kind of functional is that they can distinguish different HN type.

Proposition 3.10. *(1) If $\overrightarrow{\mu} \leq \overrightarrow{\lambda}$, then $\psi_\alpha(\sqrt{-1}\overrightarrow{\mu}) \leq \psi_\alpha(\sqrt{-1}\overrightarrow{\lambda})$ for all $\alpha \geq 1$. (2) Assume $\mu_n \geq 0$ and $\lambda_n \geq 0$. If $\psi_\alpha(\sqrt{-1}\overrightarrow{\mu}) = \psi_\alpha(\sqrt{-1}\overrightarrow{\lambda})$ for all $\alpha \geq 1$. then $\overrightarrow{\mu} = \overrightarrow{\lambda}$.*

Proof. (1) follows from [AB83], Equation 12.5. For (2), consider $f(\alpha) = \psi_\alpha(\sqrt{-1}\overrightarrow{\mu})$ and $g(\alpha) = \psi_\alpha(\sqrt{-1}\overrightarrow{\lambda})$ as functions of α. As complex valued functions, f and g clearly have analytic extensions to $\mathbb{C}\backslash\{\alpha \leq 0\}$. Suppose that $f(\alpha) = g(\alpha)$ for all $\alpha \geq 1$. Then by analyticity, $f(\alpha) = g(\alpha)$ for all $\mathbb{C}\backslash\{\alpha \leq 0\}$. If $\overrightarrow{\mu} \neq \overrightarrow{\lambda}$, then there is some k, $1 \leq k \leq n$, such that $\mu_i = \lambda_i$ for $i < k$, and $\mu_k \neq \lambda_k$; say, $\mu_k > \lambda_k$.
Then for any $\alpha > 0$:

$$\left(\frac{\mu_k}{\lambda_k}\right)^\alpha \leq \sum_{i=k}^{n}\left(\frac{\mu_i}{\lambda_k}\right)^\alpha = \sum_{i=k}^{n}\left(\frac{\lambda_i}{\lambda_k}\right)^\alpha \leq n,$$

where the middle equality follows from $f(\alpha) = g(\alpha)$ and $\mu_i = \lambda_i$ for $i < k$. Letting $\alpha \to \infty$, we obtain a contradiction. $\qquad\square$

3.3. **Approximate critical Hermitian structure.** To prove the gradient flow preserving HN type, we need equality $YMH_{\alpha,N}(A_\infty, \phi_\infty) = YMH_{\alpha,N}(\overrightarrow{\mu}_0)$. Although the weighted YMH functional is non-increasing and the HN type is non-decreasing, there still may be some jumping phenomenon illustrated in following figure
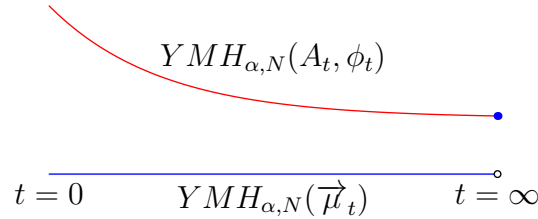
We need another tool, namely approximate critical Hermitian structure on Higgs bundle, to show there is in fact no gap between $YMH_{\alpha,N}(A_\infty, \phi_\infty)$ and $YMH_{\alpha,N}(\overrightarrow{\mu}_0)$.

The following is the Higgs version of definition introduced in [DW04] for vector bundle. Fix a Higgs bundle $(E, \bar{\partial}, \phi)$ and a Hermitian metric H. Let $\{E_i\}_{i=1}^{l}$ be the HN filtration. Associated to each E_i the

unitary projection π_i^H from E to E_i. For convenience, we set $\pi_0^H = 0$. The π_i^H are bounded L^2 Hermitian endomorphisms. Then the Harder-Narasimhan projection, $\Psi^{hn}(E, \bar{\partial}, \phi, H)$, is defined by $\sum_{i=1}^l \mu_i(\pi_i^H - \pi_{i-1}^H)$, which is a bounded L^2 Hermitian endomorphism.

Definition 3.11. *Fix $\delta > 0$ and $1 \leq p \leq \infty$. An L^p-δ-approximate critical Hermitian structure on a Higgs bundle E is a smooth metric H such that*

$$||\sqrt{-1}\Lambda\Theta_{(\bar{\partial},\phi,H)} - \Psi^{hn}(E, \bar{\partial}, \phi, H)||_{L^p} \leq \delta.$$

Theorem 3.12. *For any $\delta > 0$, there is an L^∞-δ-approximate critical Hermitian structure H on $(E, \bar{\partial}, \phi)$.*

Proof. First, by the equivalence of holomorphic structures $\bar{\partial}$ and the unitary connections A, it suffices to show that for a fixed Hermitian metric H there is a smooth complex gauge transformation g preserving the HN filtration such that:

$$(3.2) \qquad ||\sqrt{-1}\Lambda\Theta_{(g(\bar{\partial},\phi),H)} - \Psi^{hn}(g(\bar{\partial},\phi), H)||_{L^\infty} \leq \delta.$$

Next, for semistable E (i.e. the length 1 case), the result follows by the convergence $||\sqrt{-1}\Lambda\Theta_{(A_t,\phi_t)} - \mu(E)Id_E||_{L^\infty} \to 0$, where (A_t, ϕ_t) is a solution to the gradient flow Equation (2.4) with any initial condition (cf. [Don85], Cor.25, here the Higgs case is similar). With this understood, choose δ'-approximate metrics, where $0 < \delta' << \delta$, on the semistable quotients Q_i of the HN filtration of E to fix a metric H on $E = Q_1 \oplus \cdots \oplus Q_l$. Then by appropriately scaling the extension classes: $0 \to E_{i-1} \to E_i \to Q_i \to 0$, one finds a complex gauge transformation satisfying (3.2). We omit the details. $\square$

By the equivalence of L^p norm and the weighted YMH functional, we have

Corollary 3.13. *Let E be a Higgs bundle of HN type $\overrightarrow{\mu}_0$. There is $\alpha_0 > 1$ such that the following holds: given any $\delta > 0$ and any N, there is a Hermitian metric H on E such that*

$$YMH_{\alpha,N}(E, \bar{\partial}, \phi) \leq YMH_{\alpha,N}(\overrightarrow{\mu}_0) + \delta, \quad \text{for all} \ \ 1 \leq \alpha \leq \alpha_0.$$

3.4. **Proof of theorem.** With these preparation in hand, we can prove Theorem 1.2 by using the approximate critical Hermitian structure on Higgs bundle to eliminate the possibility of the jumping phenomenon,

Lemma 3.14.

$$\lim_{t\to\infty} YMH_{\alpha,N}(A_t, \phi_t) = YMH_{\alpha,N}(\overrightarrow{\mu}_0).$$

The proof is divided into two parts.

Step1, for fixed α and fixed N, define $\delta_0 > 0$ by:

(3.3)
$$2\delta_0 + \mathrm{YMH}_{\alpha,N}(\overrightarrow{\mu}_0) = \min\{\mathrm{YMH}_{\alpha,N}(\overrightarrow{\mu}) : \mathrm{YMH}_{\alpha,N}(\overrightarrow{\mu}) > \mathrm{YMH}_{\alpha,N}(\overrightarrow{\mu}_0)\},$$

where $\overrightarrow{\mu}$ runs over all possible HN types of Higgs bundles on X with the rank of $(E, \bar{\partial}, \phi_0)$. For $\overrightarrow{\mu}$ is discrete, δ_0 always exists. By corollary 3.13, consider metrics H on E with associated connection $A_0 = (\bar{\partial}, H)$ satisfying:

(3.4)
$$\mathrm{YMH}_{\alpha,N}(A_0, \phi_0) \leq \mathrm{YMH}_{\alpha,N}(\overrightarrow{\mu}_0) + \delta_0.$$

Let (A_∞, ϕ_∞) be the limit along the flow with initial condition (A_0, ϕ_0). Then combining Prop.3.5, Prop.3.10 (1), and Prop.3.8, we have:

$$\mathrm{YMH}_{\alpha,N}(\overrightarrow{\mu}_0) \leq \mathrm{YMH}_{\alpha,N}(A_\infty, \phi_\infty) \leq \mathrm{YMH}_{\alpha,N}(A_0, \phi_0) \leq \mathrm{YMH}_{\alpha,N}(\overrightarrow{\mu}_0) + \delta_0.$$

By Equation (3.3) the definition of δ_0, we must have $\mathrm{YMH}_{\alpha,N}(A_\infty, \phi_\infty) = \mathrm{YMH}_{\alpha,N}(\overrightarrow{\mu}_0)$. This shows that the result holds for initial conditions satisfying (4.4).

Step2, in the following, we want to show for any initial data, after long enough time, $\mathrm{YMH}_{\alpha,N}(A_t, \phi_t)$ will approach $\mathrm{YMH}_{\alpha,N}(\overrightarrow{\mu}_0)$ sufficient close, then reducing the problem to step1. More precisely, let us denote by (A_t^H, ϕ_t) the solution to the YMH flow at time t with initial condition $A_0 = (\bar{\partial}, H)$. We are going to prove that for any H and any $\delta > 0$, there is $T \geq 0$ such that:

(3.5)
$$\mathrm{YMH}_{\alpha,N}(A_t, \phi_t) < \mathrm{YMH}_{\alpha,N}(\overrightarrow{\mu}_0) + \delta, \quad \text{for all} \ \ t \geq T.$$

Without loss of generality, assume $0 < \delta \leq \delta_0/2$. Let $\mathcal{H}_\delta$ denote the set of smooth hermitian metrics H on E with the property that (3.5) holds for (A_t^H, ϕ) and some T. We employ open and closeness argument to show $\mathcal{H}_\delta$ containing all the smooth Hermitian metric.

First, $\mathcal{H}_\delta$ is non-empty. Indeed, any metric satisfying Equation (3.4) is in $\mathcal{H}_\delta$, and according to Theorem 3.12, we may always find such kind of metric.

Second, $\mathcal{H}_\delta$ is open. This is an easy consequence of the continuous dependence on the initial data in finite time of the flow in the C^∞ topology.

Third, $\mathcal{H}_\delta$ is closed. Let H_j be a sequence of smooth Hermitian metrics on E such that each $H_j \in \mathcal{H}_\delta$, and suppose $H_j \to K$, in the C^∞ topology, for some metric K. We want to show that $K \in \mathcal{H}_\delta$

Since $H_j \in \mathcal{H}_\delta$, we have a sequence T_j such that for all $t \geq T_j$:

$$\mathrm{YMH}_{\alpha,N}(A_t^{H_j}, \phi_t) \leq \mathrm{YMH}_{\alpha,N}(A_{T_j}^{H_j}, \phi_{T_j}) \mathrm{YMH}_{\alpha,N}(\overrightarrow{\mu}_0) + \delta.$$

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We may find a sequence $t_j \geq T_j$, s.t. $(A_{t_j}^{H_j}, \phi_{t_j}^{H_j}) \to (A_\infty^{(1)}, \phi_\infty^{(1)})$ in L^p for all p. At another hand, $(A_{t_j}^K, \phi_{t_j}^K) \to (A_\infty^{(2)}, \phi_\infty^{(2)})$.

Using Higgs version of Donaldson's functional(cf. [Don85], or the Higgs case [LZ11]), it is not difficult to show

$$(A_\infty^{(1)}, \phi_\infty^{(1)}) = (A_\infty^{(2)}, \phi_\infty^{(2)}).$$

Thus set $(A_\infty, \phi_\infty) = (A_\infty^{(1)}, \phi_\infty^{(1)}) = (A_\infty^{(2)}, \phi_\infty^{(2)})$, then

$$\lim_{j \to \infty} \text{YMH}_{\alpha,N}(A_{T_j}^{H_j}, \phi_{T_j}) = \lim_{j \to \infty} \text{YMH}_{\alpha,N}(A_{t_j}^K, \phi_{t_j}^K) = \text{YMH}_{\alpha,N}(A_\infty, \phi_\infty).$$

Hence, for j sufficiently large:

$$\text{YMH}_{\alpha,N}(A_{t_j}^K, \phi_{t_j}^K) \leq \text{YMH}_{\alpha,N}(A_\infty, \phi_\infty) + \delta$$
$$= \lim_{j \to \infty} \text{YMH}_{\alpha,N}(A_{T_j}^{H_j}, \phi_{T_j}) + \delta \leq \text{YMH}_{\alpha,N}(\overrightarrow{\mu}_0) + 2\delta$$
$$\leq \text{YMH}_{\alpha,N}(\overrightarrow{\mu}_0) + \delta_0.$$

Therefore, $K \in \mathcal{H}_\delta$.

Since the space of smooth metrics is connected, we conclude that every metric is in $\mathcal{H}_\delta$, and (3.5) holds for all $\delta > 0$ and all metric H. In particular, we can choose $\delta \leq \delta_0$ and conclude that

$$\lim_{t \to \infty} \text{YMH}_{\alpha,N}(A_t^H, \phi_t^H) = \text{YMH}_{\alpha,N}(\overrightarrow{\mu}_0), \text{ for any } H.$$

Since the choice of N was arbitrary, the proof of Lemma is complete, and Theorem 1.2 follows.

4. CONVERGENCE TO THE GRADED OBJECT

We will finish the proof of Theorem 1.3 in this section.

The proof mainly follows the argument in the proof of Theorem 5.3 in [Wil06]. For $d_{A_j} + \phi_j + \phi_j^{*H}$ is no longer a connection, there are two major differences. One is that we should modify the Chern-Weil formula, another is using the unitary connection d_A to build the Sobolev space to get the convergence.

Theorem 1.2 already shows that the type of the Harder-Narasimhan filtration is preserved in the limit. We want to show that the destabilising Higgs sub-bundles in the Harder-Narasimhan filtration along the gradient flow also converge to the destabilising Higgs sub-bundles of the limiting Higgs pair. This fact will lead us to the construction of the holomorphic morphism between $Gr^{hns}(E, d_{A_0}'', \phi_0)$ to $(E, d_{A_\infty}'', \phi_\infty)$. In the following we use the projection $\pi : E \to E$ to denote the sub-bundle $\pi(E)$.

Proposition 4.1. *Let $\{\pi_t^{(i)}\}$ be the HN filtration of a solution (A_t, ϕ_t) to the gradient flow equations (2.4), and let $\{\pi_\infty^{(i)}\}$ be the HN filtration of the limit (A_∞, ϕ_∞). Then there exists a subsequence $\{t_j\}$ such that $\pi_{t_j}^{(i)} \to \pi_\infty^{(i)}$ in L^2 for all i.*

To prove this we need the following lemmas.

Lemma 4.2. $||d''_{A_t}(\pi_t^{(i)})||_{L^2} \to 0$ *and* $||[\phi_t, \pi_t^{(i)}]||_{L^2} \to 0$

Proof. The Chern-Weil formula (3.1) shows that
(4.1)
$$deg(\pi_t^{(i)}) = \frac{\sqrt{-1}}{2\pi} \int_X tr(\pi_t^{(i)} \Lambda(F_{A_t} + [\phi_t, \phi_t^{*H}]) - ||d''_{A_t}(\pi^{(i)})||_{L^2}^2 - ||[\phi_t, \pi_t^{(i)}]||_{L^2}^2$$

Along the finite-time flow $d_i = deg(\pi_t^{(i)})$ is fixed, therefore we can rewrite Equation (4.1)

$$||d''_{A_t}(\pi^{(i)})||_{L^2}^2 - ||[\phi_t, \pi_t^{(i)}]||_{L^2}^2$$

$$(4.2) \qquad = -d_i + \frac{\sqrt{-1}}{2\pi} \int_X tr(\pi_t^{(i)} \Lambda(F_{A_\infty} + [\phi_\infty, \phi_\infty^{*H}])$$

$$+ \frac{\sqrt{-1}}{2\pi} \int_X tr(\pi_t^{(i)} \Lambda(F_{A_t} + [\phi_t, \phi_t^{*H}] - F_{A_\infty} - [\phi_\infty, \phi_\infty^{*H}])$$

Since $F_{A_t} + [\phi_t, \phi_t^{*H}] \to F_{A_\infty} + [\phi_\infty, \phi_\infty^{*H}]$ in C^∞ topology and $\pi_t^{(i)}$ uniformly bounded in L^2 (for it is a projection) then the last term in (4.2) converges to zero. Let $\overrightarrow{\mu}$ be the HN type of $(E, d''_{A_\infty}, \phi_\infty)$. Since $(d''_{A_\infty}, \phi_\infty)$ is a critical point of YMH then we also have

$$(4.3) \qquad \frac{\sqrt{-1}}{2\pi} \int_X tr(\pi_t^{(i)} \Lambda(F_{A_\infty} + [\phi_\infty, \phi_\infty^{*H}]) \leq \sum_{k \leq rank(\pi_\infty^{(i)})} \mu_k = d_i.$$

Combining all of these results, we see that $||d''_{A_t}(\pi_t^{(i)})||_{L^2} \to 0$ and $||[\phi_t, \pi_t^{(i)}]||_{L^2} \to 0$. $\square$

In particular, this lemma shows that $||\pi_t^{(i)}||^{H^1} \leq C$ and so there exists some $\tilde{\pi}_\infty^{(i)}$ and a subsequence t_j such that $\pi_t^{(i)} \to \tilde{\pi}_\infty^{(i)}$ weakly in H^1 and strongly in L^2.

Lemma 4.3. $||d''_{A_\infty}(\pi_\infty^{(i)})||_{L^2} = 0$ *and* $||[\phi_\infty, \pi_\infty^{(i)}]||_{L^2} = 0$

Proof. For $||d''_{A_\infty}(\pi_{t_j}^{(i)})||_{L^2} \leq ||d''_{A_\infty}(\pi_{t_j}^{(i)}) - d''_{A_{t_j}}(\pi_\infty^{(i)})||_{L^2} + ||d''_{A_{t_j}}(\pi_{t_j}^{(i)})||_{L^2}$, by the continuously convergence of the gradient flow and previous lemma, $||d''_{A_\infty}(\pi_{t_j}^{(i)})||_{L^2} \to 0$. Since $\pi_{t_j}^{(i)} \to \tilde{\pi}_\infty^{(i)}$ weakly in H^1 then $||d''_{A_\infty}(\pi_\infty^{(i)})||_{L^2} = 0$. The proof of $||[\phi_\infty, \pi_\infty^{(i)}]||_{L^2} = 0$ is similar. $\square$

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This lemma implies that $\pi_\infty^{(i)}$ is indeed a ϕ_∞ invariant split Higgs bundle.

Lemma 4.4. $deg(\tilde{\pi}_\infty^{(i)}) = deg(\pi_\infty^{(i)})$.

Proof. The previous lemma and Equation (4.1) show that

$$
\begin{aligned}
deg(\tilde{\pi}_\infty^{(i)}) =& \frac{\sqrt{-1}}{2\pi} \int_X tr(\tilde{\pi}_t^{(i)} \Lambda(F_{A_\infty} + [\phi_\infty, \phi_\infty^{*H}])) \\
=& \lim_{j\to\infty} \frac{\sqrt{-1}}{2\pi} \int_X tr(\pi_{t_j}^{(i)} \Lambda(F_{A_{t_j}} + [\phi_{t_j}, \phi_{t_j}^{*H}])) \\
=& \lim_{j\to\infty} (||d''_{A_{t_j}}(\pi_{t_j}^{(i)})||_{L^2}^2 + ||[\phi_{t_j}, \pi_{t_j}^{(i)}]||_{L^2}^2) + deg(\pi_{t_j}^{(i)}) \\
=& deg(\pi_\infty^{(i)})
\end{aligned}
$$

where in the last step we use the result of Theorem 1.2 that the type of HN filtration is preserved in the limit. $\square$

Notice that $\pi_{t_j}^{(i)}$ are all orthogonal projection with constant rank, the limit $\tilde{\pi}_\infty^{(i)}$ must has the same rank. For $i = 1$, $\pi_\infty^{(1)}$ is the maximal destabilising semistable Higgs sub-bundle of $(E, d''_{A_\infty}, \phi_\infty)$, which is the unique Higgs sub-bundle of this degree and rank. Therefore $\pi_\infty^{(1)} = \tilde{\pi}_\infty^{(1)}$. Proceeding by induction on the HN filtration as in [DW04], we can show all $\pi_\infty^{(i)}$ and $\tilde{\pi}_\infty^{(i)}$ are the same, then Proposition 4.1 follows. This means that not only the type of HN filtration but also the HN filtration itself is preserved by the gradient flow.

The same argument applies to the Seshadri filtration of a semistable Higgs bundle, except that because of the lack of uniqueness of the Seshadri filtration we can only conclude that the degree and rank of the limiting sub-bundle are the same. There must be another way to identify the stable Higgs bundles in the HNS filtration and $(E, d''_{A_\infty}, \phi_\infty)$.

Fix S to be the first term in the Harder-Narasimhan-Seshadri filtration of (E, d''_{A_0}, ϕ_0). Following Donaldson[Don85], if there is a nontrivial holomorphic map from S to $(E, d''_{A_\infty}, \phi_\infty)$, we can apply the basic principle that a nontrivial holomorphic map between stable bundles of the same rank and degree must be an isomorphism. Denote (A_{t_j}, ϕ_{t_j}) by (A_j, ϕ_j), and let g_j be the complex gauge transformation such that $(A_j, \phi_j) = g_j(A_0, \phi_0)$. Let $f_0 : S \to E$ be the ϕ_0-invariant holomorphic inclusion, define the map $f_j : S_j \to E$ by $f_j = g_j \circ f_0$. It is easy to check that f_j is a ϕ-invariant holomorphic bundle map from (S, d''_{A_0}, ϕ_0) to (E, d''_{A_j}, ϕ_j), here ϕ-invariant means $f_j \circ \phi_0 = \phi_j \circ f_j$. Then we have

Lemma 4.5. *Up to a subsequence, f_j converges in C^∞ to some nonzero ϕ-invariant holomorphic map f_∞.*

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Reader can refer to [LZ11] for the details of the proof.

If π_j denotes the projection to $f_j(S)$, then as mentioned under the proof of Proposition 4.1, $\pi_j \to \pi_\infty$ weakly in H^1 and strongly in L^2, where π_∞ is a subbundle of the same rank and degree as S. Denote $S_\infty = \pi_\infty(E)$, then f_∞ is in effect a map from (S, d''_{A_0}, ϕ_0) to $(S_\infty, d''_{A_\infty}, \phi_\infty)$.

A prior, $f_\infty : S \to S_\infty$ could be any bad, but we have the following lemma completely analogous to the proof of (V.7.11) in [Kob87] for holomorphic bundles and so the proof is omitted.

Lemma 4.6. *Let $(S_1, \bar{\partial}_1, \phi_1)$ be a stable Higgs bundle, and let $(S_2, \bar{\partial}_2, \phi_2)$ be a semistable Higgs bundle over a compact Riemann surface X. Also suppose that $\frac{deg(S_1)}{rank(S_1)} = \frac{deg(S_2)}{rank(S_2)}$, and let $f : S_1 \to S_2$ be a holomorphic map satisfying $f \circ \phi_1 = \phi_2 \circ f$. Then either $f = 0$ or f is injective.*

For the gradient flow preserve HN filtration, $(S_\infty, d''_{A_\infty}, \phi_\infty)$ still lies in the maximal destablishing semistable Higgs subbundle and has the highest slope in the HN filtration. Hence $(S_\infty, d''_{A_\infty}, \phi_\infty)$ can not admit any subbundle with higher slope, must be semi-stable. By above lemma, f_∞ is injective. But S_∞ has the same rank with S, then f_∞ must be an isomorphism, and $S_\infty = f_\infty(S)$ is a stable factor in the split Higgs bundle $(E, d''_{A_\infty}, \phi_\infty)$. Relabel f_∞ as $f_\infty^{(1)(1)}$ to represent the highest factor in the HNS filtration. Now we construct a isomorphism form S in $Gr^{hns}(E, d''_{A_0}, \phi_0)$ to S_∞ in $(E, d''_{A_\infty}, \phi_\infty)$. To build the entire isomorphism from $Gr^{hns}(E, d''_{A_0}, \phi_0)$ to $(E, d''_{A_\infty}, \phi_\infty)$, we make induction on the length of the HNS filtration. Let $Q = E/S$, we have $Gr^{hns}(E, d''_{A_0}, \phi_0) = S \oplus Gr^{hns}(E, d''_{A_0}{}^Q, \phi_0^Q)$, and $(E, d''_{A_\infty}, \phi_\infty) = S_\infty \oplus Q_\infty$. Follow the discussion in [LZ11], we can prove that $Q_\infty \cong Gr^{hns}(E, d''_{A_0}{}^Q, \phi_0^Q)$. Similarly, there is a map $f_\infty^{(1)(2)}$ from $S' \subset Q$ to $S'_\infty \subset Q$. Repeat this procedure, there exists a isomorphism $\{f_\infty^{(i)(j)}\}$ identifying $Gr^{hns}(E, d''_{A_0}, \phi_0)$ with $(E, d''_{A_\infty}, \phi_\infty)$.

Finally, the limit $\{f_\infty^{(i)(j)}\}$ exists along the flow independently of the subsequence chosen, then we complete the proof of Theorem 1.3.

Remark: The fixed Hermitian metric H on E for the YMH flow is arbitrary, but Theorem 1.3 informs us the limit does not depend on the choice of metric.

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